Gauge Invariant Extension of Linearized Horava Gravity

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Abstract

In the present paper we have constructed a gauge invariant extension of a generic Horava Gravity (HG) model (with quadratic curvature terms) in linearized version in a systematic procedure. No additional fields are introduced. The linearized HG model is explicitly shown to be a gauge fixed version of the Einstein Gravity (EG) thus proving the Bellorin-Restuccia conjecture in a robust way. In the process we have explicitly computed the correct Hamiltonian dynamics using Dirac Brackets appearing from the Second Class Constraints present in the HG model. We comment on applying this scheme to the full non-linear HG.

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Introduction and summary of our work: It is well known that in a generic quantum field theory higher derivative terms improve its ultraviolet behavior but, unfortunately, in the context of Einstein Gravity (EG) the above generates ghosts [1] thus rendering the covariant higher derivative extension unacceptable. To lift this impasse Horava [2] has proposed an ingenious idea of introducing higher derivative terms in the spatial sector only without modifying the kinetic part. The advantage of better ultraviolet behavior together with the non-appearance of ghosts in the Horava Gravity (HG) [2] however is achieved at a steep price: the full diffeomorphism invariance is replaced by foliation-preserving diffeomorphism invariance only. According to the works of [3, 4, 5, 6, 7], this loss of symmetry makes HG inconsistent as it induces a peculiar constraint structure and an extra dynamical mode of a non-canonical nature, (besides the graviton), and HG fails to match EG in low energy regime, a prerequisite of any viable extension of EG. Very recently the constraint structure of HG has been re-examined in [8, 9] and it is shown that, contrary to previous claims in [3, 4, 5, 6, 7], HG is a consistent theory but the presence of extra mode is unavoidable in the full theory. The role of the potential function has been studied in [10].

In the present work we study linearized HG

$$S = \int dt d^3x \sqrt{g} N(K_{ij}K^{ij} - \lambda K^2 + AR + BR_{ij}R^{ij} + CR^2).$$ (1)

and explicitly demonstrate the following:

(i) HG is a completely consistent constraint system having a conventional form of Second Class Constraints.

(ii) The λ, A and C terms do not play any role in linearized HG.

The above observations corroborate with [8, 9].

(iii) *Most importantly we provide a systematic way of further extending HG to a gauge invariant theory.* Interestingly enough, we recover the linearized EG modified by the B-term (1) contribution only. This form of improvement has been suggested in a heuristic way in [3] for linearized HG. We follow a general scheme developed in [11, 12] which can be applied to the full HG as well. The latter work is presently under study.

Linearized Horava Gravity: We start with the following form of HG (1),

$$S = \int dt \mathcal{L} = \int dt d^3 x \sqrt{g} N (G^{ijkl} K_{ij} K_{kl} + AR + BR_{ij} R^{ij} + CR^2)$$

$$= \int dt d^3 x \sqrt{g} N (K_{ij} K^{ij} - \lambda K^2 + AR + BR_{ij} R^{ij} + CR^2). \tag{2}$$

Here g_{ij} is the spatial metric, A, B, C are three dimension-full parameters of the theory, N is the Lapse function, R is the spatial Ricci scalar and K_{ij} is the extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N} (\partial_0 g_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{3}$$

with $N_i(x, t)$ is the Shift vector in ADM formalism [13] and the generalized De Witt metric G^{ijkl} is defined as

$$G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl}. \tag{4}$$

For $\lambda = 1$, $A = 1$ and $B = C = 0$ HG reduces to EG.

Consider the following perturbations to the metric:

$$g_{ij} = \delta_{ij} + h_{ij} \quad , \quad N = 1 + n \quad , \quad N_i = n_i. \tag{5}$$

Under these perturbations (5), the expressions for the extrinsic curvature and the Ricci curvature turn out to be

$$K_{ij} = \frac{1}{2} (\partial_0 h_{ij} - \partial_i n_j - \partial_j n_i) \quad , \quad K = \delta^{ij} K_{ij} = \frac{1}{2} (\partial_0 h - 2 \partial_i n^i),$$

$$R_{ij} = \frac{1}{2} (\partial^k \partial_i h_{jk} + \partial^k \partial_j h_{ik} - \partial^2 h_{ij} - \partial_i \partial_j h) \quad , \quad R = \partial_i \partial_j h^{ij} - \partial^2 h. \tag{6}$$

Using the above expressions () and the relation

$$\sqrt{g} R = \frac{1}{2} h_{ij} \left(-R^{ij} + \frac{1}{2} \delta^{ij} R \right) \tag{7}$$

in the action () we obtain the Lagrangian density $\mathcal{L}$ of second order in h:

$$\mathcal{L} = \frac{1}{4} [\partial_0 h_{ij} \partial_0 h^{ij} - \lambda (\partial_0 h)^2 - 4 (\partial_0 n_i)(\partial_j h^{ij} - \lambda \partial^i h) + (4\lambda - 2) n_i (\partial^i \partial^j n_j) - 2 n_i \partial^2 n^i]$$

$$+ \frac{1}{4} h_{ij} (\partial^2 h^{ij} - 2 \partial_k \partial^i h^{jk} + 2 \partial^i \partial^j h - \delta^{ij} \partial^2 h) + An(\partial_i \partial_j h^{ij} - \partial^2 h) + C(\partial_i \partial_j h^{ij} - \partial^2 h)(\partial_k \partial_l h^{kl} - \partial^2 h)$$

$$+\frac{B}{4}(\partial^k\partial_i h_{jk} - \partial^2 h_{ij} + \partial^k\partial_j h_{ik} - \partial_i\partial_j h)(\partial_l\partial^i h^{jl} - \partial^2 h^{ij} + \partial_l\partial^j h^{il} - \partial^i\partial^j h). \tag{8}$$

In the above expression (), we used the following notation:

$$h = \delta^{ij}h_{ij} \quad , \quad \partial^2 = \partial_i\partial^i = \delta^{ij}\partial_i\partial_j.$$

Hamiltonian of linearized Horava Gravity: From this Lagrangian () we obtain the conjugate momenta as

$$p \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 n)} = 0 \quad , \quad p^i \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 n_i)} = -(\partial_j h^{ij} - \lambda\partial^i h), \tag{9}$$

$$\pi^{ij} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 h_{ij})} = \frac{1}{2}(\partial_0 h^{ij} - \delta^{ij}\lambda(\partial_0 h)). \tag{10}$$

Taking the trace of the relation (10), we can write $\partial_0 h_{ij}$ in terms of π_{ij} as

$$\partial_0 h^{ij} = 2\left(\pi^{ij} + \frac{\lambda}{1-3\lambda}\delta^{ij}\pi\right). \tag{11}$$

Using the relations (9), (10), (11) in () we get the Hamiltonian density,

$$\mathcal{H} = p^i(\partial_0 n_i) + \pi^{ij}(\partial_0 h_{ij}) - \mathcal{L}$$

$$= \pi_{ij}\pi^{ij} - \frac{\lambda}{3\lambda-1}\pi^2 - \frac{1}{2}(2\lambda-1)n_i(\partial^i\partial^j n_j) + \frac{1}{2}n_i\partial^2 n^i$$

$$-\frac{1}{4}h_{ij}(\partial^2 h^{ij} - 2\partial_k\partial^i h^{jk} + 2\partial^i\partial^j h - \delta^{ij}\partial^2 h) - An(\partial_i\partial_j h^{ij} - \partial^2 h) - C(\partial_i\partial_j h^{ij} - \partial^2 h)(\partial_k\partial_l h^{kl} - \partial^2 h)$$

$$-\frac{B}{4}(\partial^k\partial_i h_{jk} - \partial^2 h_{ij} + \partial^k\partial_j h_{ik} - \partial_i\partial_j h)(\partial_l\partial^i h^{jl} - \partial^2 h^{ij} + \partial_l\partial^j h^{il} - \partial^i\partial^j h). \tag{12}$$

Constraint analysis a la Dirac: We now perform a standard Hamiltonian constraint analysis using the Dirac formalism [14], which we discuss below in brief.

From a given Lagrangian, one starts by computing the conjugate momentum $p = \frac{\partial L}{\partial\dot{q}}$ of a generic variable q and identifies the relations that do not contain time derivatives as (Hamiltonian) constraints. A constraint is classified as First Class Constraint (FCC) when it commutes with all other constraints and the set of constraints which do not commute are called Second Class Constraints (SCC). First Class constraints generate gauge invariance. For systems containing Second Class constraints, one has to replace the Poisson brackets

by Dirac brackets to properly incorporate the Second Class constraints. If $(\{\psi^i_\rho, \psi^j_\sigma\}^{-1})$ is the (ij)-th element of the inverse constraint matrix where $\psi^i(q, p)$ is a set of Second Class constraints, then the Dirac bracket between two generic variables $\{A(q, p), B(q, p)\}_{DB}$ is given by

$$\{A, B\}_{DB} = \{A, B\} - \{A, \psi^i_\rho\}(\{\psi^i_\rho, \psi^j_\sigma\}^{-1})\{\psi^j_\sigma, B\}, \tag{13}$$

where $\{\ ,\ \}$ denotes Poisson brackets. In this framework, the constraints ψ^i_ρ are "strongly" zero since they commute with any generic variable A: $\{A, \psi^i_\rho\}_{DB} = \{\psi^i_\rho, A\}_{DB} = 0$.

From the Lagrangian (), we get four primary constraints

$$\psi \equiv p, \tag{14}$$

$$\phi^i \equiv p^i + \partial_j h^{ij} - \lambda \partial^i h. \tag{15}$$

Requiring the time persistence of the above four constraints (14), (15) we get the following four secondary constraints

$$\chi_1 \equiv \dot{\psi} = \left\{\psi, \int \mathcal{H}(y) d^3 y\right\} = \partial_i \partial_j h^{ij} - \partial^2 h, \tag{16}$$

$$\eta^i \equiv \dot{\phi}^i = \partial_j \pi^{ij} + \frac{1}{2}(2\lambda - 1)\partial^i \partial^j n_j - \frac{1}{2}\partial^2 n^i. \tag{17}$$

Time persistence of the constraint (17) is trivially satisfied, i.e. $\dot{\eta}^i = 0$. However, $\dot{\chi}_1$ yields a tertiary constraint

$$\chi_2 \equiv \dot{\chi}_1 = \partial_i \partial_j \pi^{ij} - \frac{\lambda - 1}{3\lambda - 1}\partial^2 \pi. \tag{18}$$

The chain stops since the constraints χ_1, χ_2 (in (16) and (18) respectively) constitute a pair of SCC. The constraints ψ, ϕ^i, η^j (in (14), (15) and (17) respectively) are FCC. *Clearly we do not find any abnormality in the constraint structure as claimed earlier in* [3, 4, 5, 6, 7]. A quick count of the dynamical degrees of freedom

$$No.\ of\ dynamical\ d.o.f\ = (total\ No.\ of\ d.o.f\ ) - (2 \times\ No.\ of\ FCC)\ - (No.\ of\ SCC)$$

$$= 20 - (2 \times 7) - 2 = 4, \tag{19}$$

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shows that the system has 4 d.o.f. in phase space that correctly represent the graviton. Regarding the comment in [7] about the viability of perturbative analysis in obtaining the number of d.o.f. we believe our results are indeed valid in the present weak gravity regime.

Let us choose the gauge $n_i = 0$, (as is customary). This does not affect the bracket structure of the remaining d.o.f. The constraints (17) simplifies to

$$\eta^i \equiv \partial_j \pi^{ij}. \tag{20}$$

Using the constraints (20) the tertiary constraint (18) becomes

$$\chi_2 \equiv \pi, \tag{21}$$

provided $\lambda \neq 1$.

Using the constraints (16), (21) and the gauge $n_i = 0$, the Hamiltonian () now reduces to

$$\mathcal{H} = \pi_{ij}\pi^{ij} - \frac{1}{4}h_{ij}(\partial^2 h^{ij} - 2\partial_k\partial^i h^{jk} + 2\partial^i\partial^j h - \delta^{ij}\partial^2 h)$$
$$-\frac{B}{4}(\partial^k\partial_i h_{jk} - \partial^2 h_{ij} + \partial^k\partial_j h_{ik} - \partial_i\partial_j h)(\partial_l\partial^i h^{jl} - \partial^2 h^{ij} + \partial_l\partial^j h^{il} - \partial^i\partial^j h). \tag{22}$$

Let us pause to note the following point: As we had advertised in the first section, the parameters A, C are absent in the weak limit but the role of λ is interesting. Apparently λ has disappeared from consideration but it still has a a non-trivial effect in an indirect way: the present constraint structure of HG having both FCC and SCC is distinct from EG constraint structure (the latter having only FCC). As mentioned below (18), in HG we consider $\lambda \neq 1$ otherwise for $\lambda = 1$ the constraint structure collapses to only FCCs that is true for EG.

Dirac Brackets and Hamiltonian dynamics: For the two SCCs, χ_1 and χ_2 ((16), (21) respectively), the constraint matrix and its inverse are given below:

$$\{\chi_1(x) , \chi_2(y)\} = \begin{pmatrix} 0 & -2\partial^2\delta^3(x-y) \\ 2\partial^2\delta^3(x-y) & 0 \end{pmatrix}$$

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$$\{\chi_1(x) \ , \ \chi_2(y)\}^{-1} = \begin{pmatrix} 0 & \frac{1}{2\partial^2}\delta^3(x-y) \\ -\frac{1}{2\partial^2}\delta^3(x-y) & 0 \end{pmatrix}. \tag{23}$$

Using () in the definition of the Dirac brackets (13), we have

$$\{h_{ij}(x), \pi^{kl}(y)\}_{DB} = \frac{1}{2}\left(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l - \delta_{ij}\delta^{kl} + \delta_{ij}\frac{\partial^k\partial^l}{\partial^2}\right)\delta^3(x-y),$$

$$\{h_{ij}(x), h_{kl}(y)\}_{DB} = \{\pi^{ij}(x), \pi^{kl}(y)\}_{DB} = 0. \tag{24}$$

Clearly the mixed bracket has a non-canonical structure.

It is straightforward to exploit the Dirac bracket () to compute the Hamiltonian equations of motion:

$$\dot{h}_{ij} = \left\{h_{ij}, \int d^3y \mathcal{H}(y)\right\}_{DB} = 2\pi_{ij} + \delta_{ij}\frac{\partial^k\partial^l}{\partial^2}\pi_{kl}. \tag{25}$$

$$\dot{\pi}_{ij} = \left\{\pi_{ij}, \int d^3y \mathcal{H}(y)\right\}_{DB}$$

$$= \frac{1}{2}\left[\partial^2 h_{ij} - \partial^k\partial_i h_{jk} - \partial^k\partial_j h_{ik} + \partial_i\partial_j h + B\left(\partial^2(\partial^2 h_{ij}) - \partial^2\partial^k\partial_i h_{jk} - \partial^2\partial^k\partial_j h_{ik} + \partial^2\partial_i\partial_j h\right)\right]. \tag{26}$$

Finally we recover the equation of motion for h_{ij},

$$\Box h_{ij} = \partial^k\partial_i h_{jk} + \partial^k\partial_j h_{ik} - \partial_i\partial_j h + B(\partial^2\partial^k\partial_i h_{jk} + \partial^2\partial^k\partial_j h_{ik} - \partial^2\partial_i\partial_j h - (\partial^2)^2 h_{ij}), \tag{27}$$

where

$$\Box h_{ij} = -\ddot{h}_{ij} + \partial^2 h_{ij}.$$

Taking the trace of the above equation (27) and imposing the constraint (16) we have

$$\ddot{h} = 0, \tag{28}$$

which clearly shows that the scalar h does not propagate at all in case of the HG model.

Hence, together with (28), (27) correctly represents the dynamics of a spin-2 field h_{ij}, modified by the higher derivative terms. In case of EG theory, the dynamics of the spin-2 field is also governed by the same equation (27), hence non-propagation of the scalar h holds

true for EG theory. This constitutes first part of our work that is proving the consistency of the (linearized) HG. Our result is consistent with [15, 16].

Gauge invariant extension of linearized HG: Gauge invariant theories are ubiquitus in modern quantum field theoretic framework. Gauge invariance plays an essential role in the quantization programme. In the prsent context of HG, the weaknesses of the theory principally result from a loss of gauge invariance, (i.e. loss of full diffeomorphism invariance). As we have explicitly demonstrated above, HG with SCC in reduced space is indeed a consistent theory but quantization of the resultant non-canonical Dirac Bracket algebra given in () can be problematic. It would be very convenient if one can construct a gauge invariant analogue of the HG. Precisely this task will be performed in this section for the linearized HG. This requires conversion of the mixed SCC - FCC system to a pure FCC system or more explicitly we wish to modify the linearized HG with the SCC pair χ_1, χ_2 to a gauge invariant theory with one FCC, constructed out of the pair χ_1, χ_2. For a generic theory, such as HG having a complicated constraint structure, the above mentioned task is quite formidable. Fortunately there is a tailor-made scheme formulated originally by Mitra and Rajaraman [11] and further developed by Vytheeswaran [12]. The idea is to interpret the original gauge-non-invariant theory (with two SCCs) as a gauge fixed version of the (to be constructed) gauge-invariant theory with one FCC. The scheme has been termed as Gauge Unfixing (GU) [11, 12]. If there are additional FCCs they remain intact on the constraint surface.

The construction proceeds as follows. From a generic pair of SCC χ_1, χ_2 choose arbitrarily one of them, say χ_1, construct the combination,

$$\chi_1 \to \chi \equiv \{\chi_1, \chi_2\}^{-1} \chi_1. \tag{29}$$

This allows one to ignore χ_2 and consider χ as the single FCC. This means that one needs to modify the system such that χ turns out to be an FCC. A mapping of any generic

variable A to its gauge invariant extension A_{GU} is given by,

$$A_{GU} \equiv A - \chi_2\{\chi, A\} + \frac{1}{2!}\chi_2^2\{\chi, \{\chi, A\}\} - \ldots\ldots \quad (30)$$

Note that

$$\{\chi, A_{GU}\} = 0$$

by construction. Thus the extended Hamiltonian system has one FCC χ. One can trivially revert back to the original system by choosing the gauge χ_2. But clearly the new system allows much more freedom in choosing any other convenient gauge.

Let us apply this Gauge Unfixing method to the HG model. We rescale one of the SCC pair χ_1 in (16) to χ:

$$\chi(x) \equiv \int d^3y \{\chi_1(x), \chi_2(y)\}^{-1}\chi_1(y) = \frac{1}{2}\left(h(x) - \frac{\partial_i\partial_j}{\partial^2}h^{ij}(x)\right). \quad (31)$$

Using the new FCC χ (31), we compute the gauge invariant extension of (or Gauge Unfixed) Hamiltonian $\mathcal{H}_{GU}$ as

$$\mathcal{H}_{GU}(x) \equiv \mathcal{H}(x) - \int \pi(y)\{\chi(y), \mathcal{H}(x)\}d^3y + \frac{1}{2}\int \pi(y)\pi(z)\{\chi(z), \{\chi(y), \mathcal{H}(x)\}\}d^3yd^3z$$

$$= \mathcal{H}(x) - \pi^2(x) + \frac{1}{2}\pi^2(x)$$

$$= \pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 - \frac{1}{4}h_{ij}(\partial^2 h^{ij} - 2\partial_k\partial^i h^{jk} + 2\partial^i\partial^j h - \delta^{ij}\partial^2 h)$$

$$-\frac{B}{4}(\partial^k\partial_i h_{jk} - \partial^2 h_{ij} + \partial^k\partial_j h_{ik} - \partial_i\partial_j h)(\partial_l\partial^i h^{jl} - \partial^2 h^{ij} + \partial_l\partial^j h^{il} - \partial^i\partial^j h). \quad (32)$$

It is interesting to note that this Gauge Unfixed Hamiltonian () is exactly the (B-term dependent) spatial higher derivative extension of EG Hamiltonian. As we have already remarked, the higher derivative A and C terms do not contribute to the linearized theory. This is not entirely unexpected since *manifestly* the symmetry violating parameter λ was absent in the reduced HG model (), (). This concludes the second part of our work: constructing a gauge invariant extension of the linearized HG. Obviously we have not constructed a new theory, (that was not the intention), but the importance of our work lies in explicitly

showing that there is a systematic way, (Gauge Unfixing method of [11, 12]), by which one can correctly reproduce a gauge invariant extension of as involved a field theory as the HG. Our work strengthens further the result of [8, 9] that linearized HG is equivalent to (higher derivative) EG.

Discussion: Since we have already summarized our results in the beginning let us conclude by mentioning the work we have undertaken now. It will be truly interesting if one can carry through this Gauge Unfixing scheme for the full Horava gravity model because it is bound to generate new non-trivial *gauge invariant* extension of the Horava model, containing explicitly the lapse dependent terms. This will surely be another "improved" version of Horava gravity. The possibility of this have already been suggested in [7, 9]. We hope the present work can act as a stepping stone for this development.

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